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DECOMPOSITION OF LINEAR PROGRAMS BY DYNAMIC PROGRAMMING

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ABSTRACT

The decomposition principle of Dantzig and Wolfe is a method for breaking large linear programs with a block diagonal structure into a set of smaller subprograms. An alternative decomposition scheme derived from a dynamic programming approach is proposed here. This results in a series of parametric linear subprograms whose recursive solution yields the solution to the original linear program.

Dantzig and Wolfe [2,* 3] have shown how linear programs of the form

$$v_r = \max \sum_{j=1}^{r} c_j X_j,$$

subject to:

$$\begin{array}{llllll}
\bar{A}_1 X_1 + \bar{A}_2 X_2 + \ldots + \bar{A}_j X_j + \ldots + \bar{A}_r X_r & = \bar{b} \\
A_1 X_1 & = b_1 \\
\quad A_2 X_2 & = b_2 \\
\qquad \ddots & \vdots \\
\qquad\quad A_j X_j & = b_j \\
\qquad\qquad \ddots & \vdots \\
\qquad\qquad\quad A_r X_r & = b_r
\end{array}$$

$$X_j \geq 0$$

can be decomposed into a set of linear subprograms each of which is of considerably smaller size than the original. In the above c_j and X_j are vectors with n_j components, $\bar{b}$ and b_j are vectors with m and m_j components, respectively, and $\bar{A}_j$ and A_j are $m \cdot n_j$ and $m_j \cdot n_j$ matrices, respectively.

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*See Ref. [2], pages 448-469.

Using the decomposition principle, one solves r linear subprograms with constraints $A_j X_j = b_j$. The objective functions for the subprograms are obtained from the original objective function and from the coupling constraints $\Sigma \bar{A}_j X_j = \bar{b}$. The overall problem is solved iteratively. That is, a set of solutions to the subprograms, together with the coupling constraints, generate new objective functions for the subprograms. These are then resolved. The iterative process is finite.

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An alternative decomposition scheme for the above linear program can be obtained from a dynamic programming approach. It has the advantage that each subprogram only need be solved once. Each subprogram is, however, a parametric linear program in the coefficients $\bar{b}$.

Consider a parametric linear subprogram of the original problem:

$$v_{r-1}(Y_{r-1}) = \max \sum_{j=1}^{r-1} c_j X_j$$

$$\begin{array}{llllllll}
\bar{A}_1 X_1 + & \bar{A}_2 X_2 + \ldots + & \bar{A}_j X_j + \ldots + & \bar{A}_{r-1} X_{r-1} & = Y_{r-1} \\
A_1 X_1 & & & & = b_1 \\
& A_2 X_2 & & & = b_2 \\
& \ddots & & & \vdots \\
& & A_j X_j & & = b_j \\
& & \ddots & & \vdots \\
& & & A_{r-1} X_{r-1} & = b_{r-1}
\end{array}$$

$$X_j \geq 0.$$

It can be shown from the theory of parametric programming [4, THEOREM 1] that v_{r-1} is a piecewise linear, concave function of each component of Y_{r-1}. The places at which v_{r-1} changes slope correspond to values of Y_{r-1} at which there is a change in basis required to maintain primal feasibility.

Suppose that v_{r-1} were known for all values of Y_{r-1} which satisfy

$$\text{(1:r)} \qquad \bar{A}_r X_r + Y_{r-1} = \bar{b}$$

$$\text{(2:r)} \qquad A_r X_r = b_r$$

$$\text{(3:r)} \qquad X_r \geq 0.$$

Then, from the dynamic programming principle of optimality [1, p. 83] it follows immediately that

(4:r) $$v_r = \max_{X_r, Y_{r-1}} \left[c_r X_r + v_{r-1}(Y_{r-1})\right]$$

subject to restrictions (1:r), (2:r), and (3:r). The solution of Eq. (4:r) requires the maximization of a piecewise linear, separable concave function, subject to linear constraints. To solve (4:r) as a linear program the piecewise linear functions are replaced by smooth linear functions, but at the expense of introducing an additional variable for each change in slope of v_{r-1}. By using the "bounded variable" method, however, it is possible to maintain the original number of basic variables [2, pp. 482-86].

Of course the subprogram from which $v_{r-1}(Y_{r-1})$ is determined can be decomposed in the same way as the original problem. Applying this decomposition scheme r times we obtain the necessary recursion relations.

$$\left.\begin{aligned}
&(4{:}j) && v_j(Y_j) = \max_{X_j, Y_{j-1}} \left[c_j X_j + v_{j-1}(Y_{j-1})\right] \\
&\text{subject to} && \\
&(1{:}j) && \bar{A}_j X_j + Y_{j-1} = Y_j \\
&(2{:}j) && A_j X_j = b_j \\
&(3{:}j) && X_j \geq 0
\end{aligned}\right\} j = 1, \ldots, r$$

with $v_0 = 0$, $Y_0 = 0$ and $Y_r = \bar{b}$.

Each of the r successive linear programs, except the first, contains $(n_j + m)$ variables and $(m_j + m)$ constraints. In each problem, however, additional variables and constraints are required to take care of the piecewise linear portions of the objective functions, though by using the bounded variable technique these will not cause an enlargement of bases. The first problem $(j = 1)$ contains n_1 variables, $(m_1 + m)$ constraints, and there are no piecewise linear functions.

For $j = 1, \ldots, r - 1$, Y_j are parameters, so a parametric linear programming algorithm must be used [4]. However, since $Y_r = \bar{b}$, the last maximization need not be parametric unless a sensitivity analysis on $\bar{b}$ is desired.

EXAMPLE

Hadley [5, pp. 407-411] has solved the following linear program using the decomposition principle of Dantzig and Wolfe. It is solved here using the dynamic programming decomposition to compare this scheme with theirs. From this one small example, however, it is hardly possible to draw any conclusions as to respective computational efficiencies.

$$\text{Max } x_1 + 8x_2 + 5x_3 + 6x_4$$

subject to:

$$x_1 + 4x_2 + 5x_3 + 2x_4 \leq 7$$

$$2x_1 + 3x_2 \leq 6$$

$$x_3 \le 4$$

$$x_4 \le 3$$

$$3x_3 + 4x_4 \ge 12$$

$$x_1, x_2, x_3, x_4 \ge 0 .$$

Let

$$v_1(y_1) = \max x_1 + 8x_2$$

subject to:

$$x_1 + 4x_2 \le y_1, \qquad 0 \le y_1 \le 7$$

$$2x_1 + 3x_2 \le 6$$

$$x_1, x_2 \ge 0 .$$

This two-dimensional parametric linear can be solved graphically to obtain $x_1 = 0$, $x_2 = y_1/4$ and $v_1(y_1) = 2y_1$, $0 \le y_1 \le 7$.

To complete the solution

$$v_2 = \max 5x_3 + 6x_4 + 2y_1 ,$$

subject to:

$$5x_3 + 2x_4 + y_1 \le 7$$

$$x_3 \le 4$$

$$x_4 \le 3$$

$$3x_3 + 4x_4 \ge 12$$

$$x_3, x_4, y_1 \ge 0 .$$

The solution to this linear subprogram is $x_3 = 0$, $x_4 = 3$, $y_1 = 1$, $v_2 = 20$. Since $x_2 = y_1/4$, the optimal solution to the original problem is $(x_1, x_2, x_3, x_4) = (0, 1/4, 0, 3)$.

CONCLUSION

For the decomposition scheme to be computationally effective the subprograms must have considerably fewer constraints and variables than the original linear program. It appears that when there are very few coupling constraints and a large number of subprograms this decomposition algorithm will be most useful. But as the number of coupling constraints increases, the method loses efficiency because of the parametric solutions required.

An important feature of the dynamic programming decomposition is the ease with which certain post-optimality problems can be handled. If the original problem is modified by the addition of another subprogram $(\bar{A}_{r+1}, A_{r+1}, c_{r+1}, X_{r+1})$, the recursion analysis is extended by the addition of only one more parametric program. On the other hand, the Dantzig-Wolfe algorithm might require several more iterations.

The theoretical simplicity of the dynamic programming decomposition leads to similar schemes for decomposing quadratic and convex programs. It is interesting that this decomposition scheme shows how several optimization methods can be synthesized to solve a large optimization problem. The dynamic programming approach decomposes a large linear program into a series of parametric subprograms and the parametric solutions lead to the maximization of piecewise linear functions in the subsequent subprograms.

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